AN ANALYTICAL INVESTIGATION OF THE TRANSIENT RESPONSE TIME OF A SIMULATED SUPERSONIC WIND TUNNEL PRESSURE INSTRUMENTATION SYSTEM

12T

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Approved:

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NOTATION AND ABBREVIATIONS

ENGLISH:

a - radius of the capillary tube

c - subscript, denotes pressure capsule

c_1 - constant

c_2 - constant

c_3 - constant

c_4 - constant

c_5 - constant

d - diameter of capillary tube; ordinary differential operator

e - natural logarithmic base

f - function of space variable

g - function of time

k - product of gas constant and absolute temperature

k_1 - constant parameter

k_2 - constant parameter

ℓ - length of capillary tubing

L - characteristic length

o - subscript, denotes initial condition

p - pressure

p_c - capsule pressure

$\bar{p}$ - characteristic pressure

p_{c_o} - initial line-capsule pressure

p_R - reservoir pressure (constant)

$\hat{p}$ - perturbation pressure

$\hat{p}_c$ - perturbation pressure in the capsule

$\hat{p}_{c_o}$ - initial perturbation pressure in the line and capsule

P - nondimensional pressure

P_c - nondimensional capsule pressure

P_{c_o} - nondimensional initial pressure in the line and capsule

$\hat{P}$ - nondimensional perturbation pressure

$\hat{P}_c$ - nondimensional perturbation pressure in the line and capsule

$\hat{P}_{c_o}$ - nondimensional perturbation pressure initially in the line and capsule

Q - rate of mass flow

R - gas constant

t - time

$\bar{t}$ - nondimensional time

T - absolute temperature

V - volume of the pressure sensing capsule

W - control volume

x - spatial position

X - nondimensional spatial position

y - coordinate

z - coordinate

GREEK:

β - eigenvalue

γ - eigenvalue

Δ - denotes incremental quantity

λ - eigenvalue

μ - coefficient of viscosity

ρ - mass density

τ - characteristic time

τ_R - response time

OTHER:

$\cdot$ - dot, superscript, denotes differentiation with respect to time

$'$ - prime, superscript, denotes differentiation with respect to spatial position

$\wedge$ - roof, superscript, denotes a perturbation quantity

ABBREVIATIONS:

lb(s).- pound(s)

in. - inch

ft. - foot, feet

sec. - second(s)

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SUMMARY

An analytical investigation of transient response time is presented for an idealized supersonic wind tunnel pressure measuring system. The pressure being measured is transmitted to a pressure sensitive device through a long, small bore capillary tube. If a difference exists between the pressure in the measuring instrument and the pressure to be measured, a finite mass of air must be evacuated from the system before accurate readings can be obtained. A finite period of time is required to accomplish this evacuation. Specifically, the response time is defined as the time required for the pressure in the measuring unit to come to within one per cent of the pressure being measured.

The theory of viscous, compressible, continuum flow through long, small bore tubing is embodied in a non-linear, second order, partial differential equation. This is the equation representing the evacuation of the measuring system. The equation is linearized for the case where the initial pressure differential is small compared to the pressure to be measured. A solution of the linearized equation is presented which satisfied the linearized boundary conditions, and roughly approximates the initial condition of a step function in pressure.

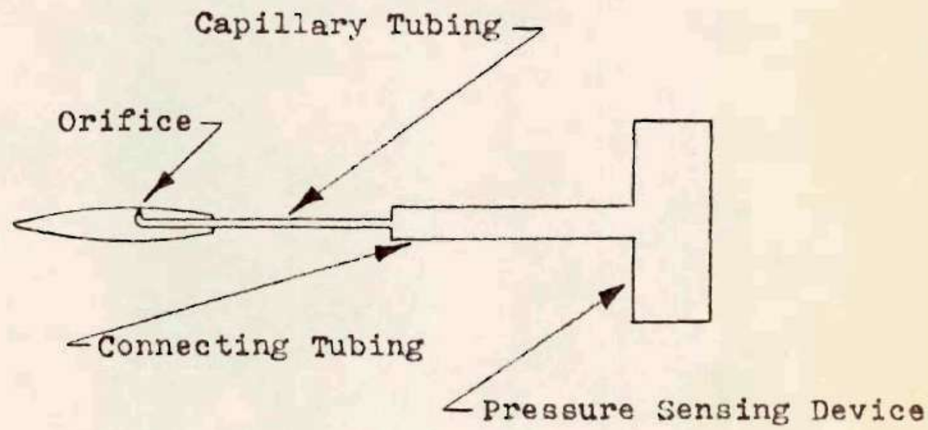
The results of this analysis are compared to experimental and analytical results of cases for which the initial step

functions in pressure are not small. When it is recalled that an initially large step function is contrary to the condition of linearization, the results appear to be quite reasonable. Nevertheless, the error is sufficiently large that the results are of qualitative interest only. Curves are also presented showing capsule pressure variation with time for the cases where the capsule pressures were initially fifty per cent greater than the reservoir pressures. With the exception of an expected time displacement towards the origin, these curves display close agreement with the segments of the experimental curves which lie in the same range of capsule pressures. These curves support the belief that the linear equations can be applied with reasonable accuracy if the step functions in pressure are initially small.

CHAPTER I

INTRODUCTION

General -- Modern high speed aircraft and missile development requires that extensive pressure measurements be made in supersonic wind tunnels for purposes of evaluating load distributions and certain aerodynamic coefficients. A pressure measuring system typical of those used in many high speed tunnel installations is shown schematically in Figure 1. Such a system consists of an orifice, a capillary tube, a connecting tube, and a pressure sensitive device remotely located from the place where the pressure is to be measured. If a pressure differential exists between the model surface and the pressure sensitive device, then a certain amount of air must be evacuated from the system before the pressure in the sensing instrument approaches that on the model surface. Such a pressure differential occurs when the system is opened to a running wind tunnel. The time required for the pressure in the sensing unit to adjust to within one per cent of the pressure to be measured is defined as the response time of the system. This response time can easily be of such magnitude so as to be intolerable in the case of the intermittently operating tunnel, and it may seriously restrict the number of readings that may be taken during a given period of continuous type operation. In either case the system should be designed for the minimum response time allowed by the physical limitations of the test set up.



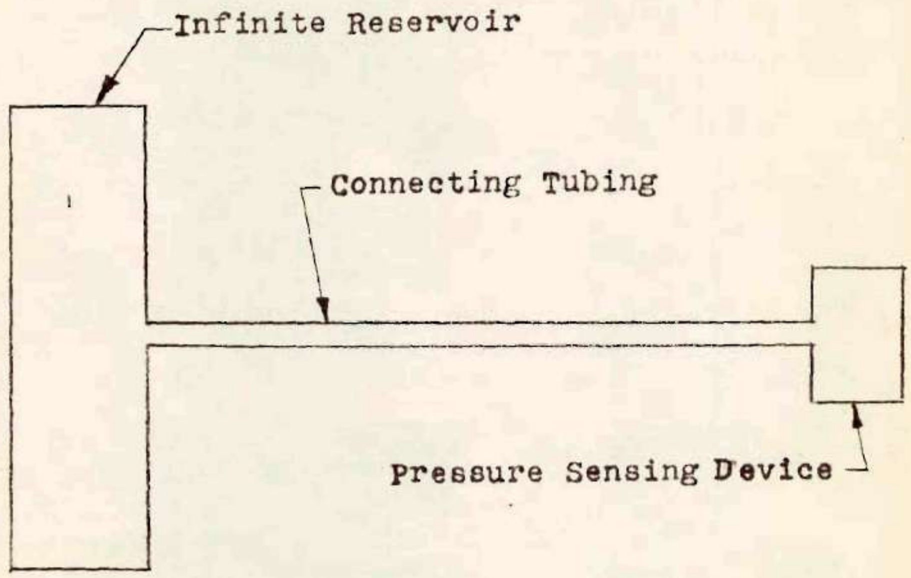
Typical Pressure Measuring System

Figure 1

The idealized system. -- The major parameters governing the magnitude of the response time are the lengths and diameters of the capillary and connecting tubes, the orifice diameter, the volume of the pressure sensitive device, the end pressures and the coefficient of viscosity. Neither the orifice nor the discontinuity in tube diameter (i.e., at the juncture of the capillary and connecting tubes) can be handled conveniently by analytical methods. To avoid this difficulty, the system shown in Figure 1 is reduced to an idealized case (shown in Figure 2) for which the connecting tube has zero length and for which the orifice has the diameter of the capillary tubing. The pressure is constant on the surface of the model during a given run; the capillary tube is then assumed to open into an infinite reservoir.

Previous investigations of the idealized system. -- Kendall (1) conducted an analytical investigation for the idealized system which used both constant and variable volumes of the pressure sensitive element. This particular analysis did not take into account the initial transients, hence application of the results are limited to the quasi-steady processes having fairly large response times.

Probably the most significant contribution which accounted for the initial transients is the investigation of Ducoffe (2). This work included extensive time lag experiments for both systems shown in Figures 1 and 2 as well as numerical solutions which, by comparison with experimental results, demonstrate the validity of the theory as applied to the idealized configuration. Ducoffe's entire investigation was conducted with pressure sensitive devices having essentially constant volumes.



Idealized Pressure Measuring System

Figure 2

It is the object of this paper to present a linearized analysis of the response time of the idealized system having a constant volume pressure sensing unit, and to discuss the application of the solution to the cases where the linearized equations are initially inapplicable. This analysis cannot be expected to accurately predict the response time for a system if the orifice is small compared to the diameter of the capillary tube, or if the connecting tube length is excessive, or if the diameter of the connecting tube differs radically from that of the capillary tube. For these cases, however, the analysis gives a qualitative insight into the effects of the other important parameters during the stabilization process.

CHAPTER II

FLOW EQUATIONS

General. -- The equation of motion and the associated boundary and initial conditions for the system shown in Figure 2 are derived in a manner similar to that of Ducoffe (3) and Kendall (4). In order to obtain a mathematical solution in closed form the equation of motion and boundary conditions are linearized and solved.

In the following derivation only the condition of continuum flow is considered. It is assumed that the flow process takes place in a small diameter tube of constant cross section, and that the flow process is isothermal. The assumption of an isothermal flow process is justified by the following argument. Since the tubing under consideration is thin walled and of small diameter, the ratio of surface area to enclosed volume is relatively large. Such being the case, if a temperature differential exists across the tube walls, heat transfer takes place. Furthermore, dissipation of kinetic energy into heat energy occurs by virtue of the viscous shearing forces at the walls of the tube. Immediately after the fluid is accelerated (i.e. as it enters the tube), with the accompanying decrease of pressure and temperature, the two effects just described tend to restore the temperature to its initial value. The latter effect, while probably the smaller of the two, is believed to be sufficiently large to be included.

Derivation of the flow equation.--To facilitate mathematical treatment of the problem, the condition of laminar, fully developed flow is considered to represent the process at any instant of time. The Hagen-Poiseulle law (5) is thus regarded as satisfying the above condition, and is expressed by

$$Q = -\pi \frac{a^4}{8\mu} \rho \frac{\partial p}{\partial x} , \tag{1}$$

where Q is the rate of mass flow, a is the radius of the tube, μ is the coefficient of viscosity, ρ is the mass density, p is the pressure, and x the spatial coordinate along the tube axis.

The equation of state for an isothermal process is

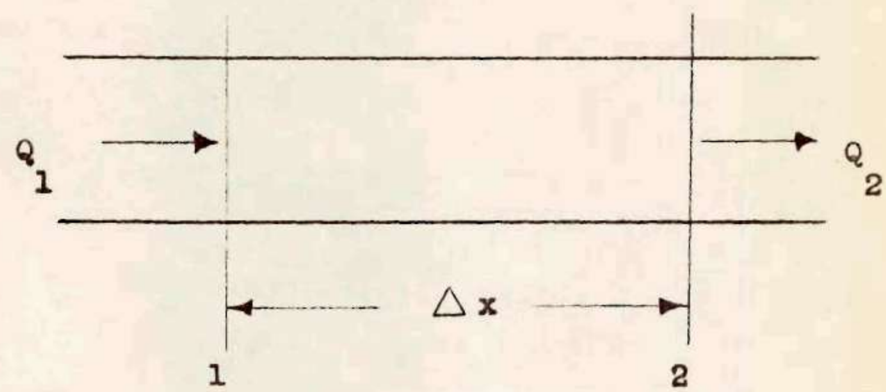
$$p = k\rho , \tag{2}$$

where k = RT, R being the gas constant and T the absolute temperature. By substituting Equation (2) into Equation (1), the rate of mass flow becomes

$$Q = -\pi \frac{a^4}{8\mu} \frac{p}{k} \frac{\partial p}{\partial x} . \tag{3}$$

In Figure 3 is shown a section of the tube of length Δx. The net rate of mass flow into the system shown in Figure 3 must be equal to the rate of change of mass enclosed. This is expressed by the divergence theorem in the form

$$Q_1 - Q_2 = \iiint_W \frac{\partial \rho}{\partial t} dW , \tag{4}$$



Mass Flow Through Capillary Tube

Figure 3

where W is the volume enclosed and t is the time. This may be written as

$$Q_1 - Q_2 = \pi a^2 \int_{x_1}^{x_2 = x_1 + \Delta x} \frac{\partial \rho}{\partial t} dx \ . \tag{5}$$

Since Δx is small, the spatial density variation within the Δx range is of higher order and is neglected; therefore at any instant $\frac{\partial \rho}{\partial t}$ may be moved to the outside of the integral in Equation (5), which now becomes

$$Q_1 - Q_2 = \pi a^2 \Delta x \frac{\partial \rho}{\partial t} \ . \tag{6}$$

By replacing the Q's with their equivalents given by Equation (3), Equation (6) becomes

$$\frac{\pi a^4}{8\mu k}\left[\frac{\left(p \frac{\partial p}{\partial x}\right)_{x+\Delta x} - \left(p \frac{\partial p}{\partial x}\right)_x}{\Delta x}\right] = \pi a^2 \frac{\partial \rho}{\partial t} \ , \tag{7}$$

which, in the limit as $\Delta x \rightarrow 0$, reduces to

$$\frac{a^2}{8\mu k} \frac{\partial}{\partial x}\left(p \frac{\partial p}{\partial x}\right) = \frac{\partial \rho}{\partial t} \ . \tag{8}$$

With the isothermal condition, Equation (8) is

$$\frac{a^2}{8\mu} \frac{\partial}{\partial x}\left(p \frac{\partial p}{\partial x}\right) = \frac{\partial p}{\partial t} \ . \tag{9}$$

Initial and boundary conditions.--The x-origin is taken at the pressure sensing end and $x = \ell$ is taken at the model or reservoir end of the system, where ℓ is the length of the tubing. The pressure release valve which opens the system to the reservoir is placed immediately upstream of the reservoir end. The initial condition is then

$$p = \text{constant} = p_{c_0} \quad (0 \leq x < \ell) \ , \tag{10}$$

where p_{c_0} is the initial pressure in both the tube and pressure capsule.

The boundary condition at $x = \ell$ is

$$p = \text{constant} = p_R \ , \tag{11}$$

where p_R is the pressure of the reservoir (i.e. at the model surface).

The boundary condition at x=0 is derived from continuity considerations in that the rate of change of mass within the sensing unit must equal the flux of mass entering the tubing. This condition is expressed by

$$Q\Big]_{x=0} = -\frac{\pi a^4}{8\mu k}\rho\frac{\partial p}{\partial x}\Big]_{x=0} = -\frac{d}{dt}(\rho V)\Big]_{x=0} \ , \tag{12}$$

where V is the fluid volume enclosed by the sensing unit. Typical pressure capsules have small volumes, with relatively large ratios of surface area to enclosed volume. Hence it is assumed that heat transfer through the walls of the capsule maintains the original temperature level. Equation (2) is then applicable. Volume change (6) of the capsule because of diaphragm deflection (or its equivalent) is usually quite small and is neglected in this analysis. Equation (12) then reduces to

$$\frac{\pi a^4}{8\mu V} p \frac{\partial p}{\partial x}\Big]_{x=0} = \frac{\partial p}{\partial t}\Big]_{x=0} . \tag{13}$$

Dimensionless form of the equation of motion and the initial and boundary conditions.-- In order to further simplify the problem and to isolate similarity parameters, it is necessary to nondimensionalize the set of equations. Accordingly, nondimensional variables P, X, and $\bar{t}$ are defined by the following relations,

$$P = \frac{p}{\bar{p}} , \tag{14}$$

$$X = \frac{x}{L} , \tag{15}$$

and

$$\bar{t} = \frac{t}{\tau} , \tag{16}$$

where $\bar{p}$, L, and τ are a characteristic pressure, a characteristic length, and a characteristic time, respectively. The proper choice of these characteristic quantities is discussed later.

The equation of motion, when written in the form

$$\frac{a^2}{8\mu}\left[p\frac{\partial^2 p}{\partial x^2} + \left(\frac{\partial p}{\partial x}\right)^2\right] = \frac{\partial p}{\partial t} , \tag{17}$$

becomes, by substitution of Equations (14), (15), and (16),

$$\frac{\partial P}{\partial \bar{t}} = \frac{1}{8}\left(\frac{a^2 \bar{p}\, \tau}{L^2 \mu}\right)\left[P \frac{\partial^2 P}{\partial X^2} + \left(\frac{\partial P}{\partial X}\right)^2\right]. \tag{18}$$

Similarly, the nondimensional boundary conditions are

$$\left.\frac{\partial P}{\partial \bar{t}}\right]_{X=0} = \frac{\pi}{8}\left(\frac{a^2 \bar{p}\, \tau}{L^2 \mu}\right)\left(\frac{L a^2}{V}\right) P \left.\frac{\partial P}{\partial X}\right]_{X=0}, \tag{19}$$

and

$$P\Big]_{X=\frac{\ell}{L}} = \left.\frac{p_R}{\bar{p}}\right]_{X=\frac{\ell}{L}}, \tag{20}$$

where ℓ is the tubing length.

The nondimensional initial condition is

$$P = \frac{p_{c_0}}{\bar{p}} \quad \left(0 \le X < \frac{\ell}{L}\right), \qquad P = \frac{p_R}{\bar{p}}, \ X = \frac{\ell}{L}. \tag{21}$$

The terms $\frac{a^2}{L^2}\frac{\bar{p}\,\tau}{\mu}$ and $\frac{La^2}{V}$ are similarity parameters. The former is termed a dynamic parameter and the latter a geometric one. The major parameters affecting the flow are the inside diameter of the tube, d, the length of the tube, ℓ, the volume of the pressure measuring unit, V, the absolute pressure at the model surface, p_R, and the viscosity coefficient of the fluid, μ. The time most characteristic of the system is the response time. Accordingly, the characteristic quantities are defined as

$$\tau = \tau_R, \tag{22}$$

$$L = \ell , \tag{23}$$

and

$$\bar{p} = p_R , \tag{24}$$

where τ_R is the response time. The response time was previously defined as the time required for the pressure at the measuring end to adjust to within one per cent of the reservoir absolute pressure.

Substitution of Equations (22), (23), and (24) into Equations (18), (19), (20), and (21) yields:
Equation of motion,

$$\frac{\partial P}{\partial \bar{t}} = \frac{1}{8}\left(\frac{a^2}{\ell^2}\,\frac{p_R\,\tau_R}{\mu}\right)\left[P\,\frac{\partial^2 P}{\partial X^2} + \left(\frac{\partial P}{\partial X}\right)^2\right] . \tag{25}$$

Boundary condition at X = 0,

$$\frac{\partial P}{\partial \bar{t}} = \frac{\pi}{8}\left(\frac{a^2}{\ell^2}\,\frac{p_R\,\tau_R}{\mu}\right)\left(\frac{\ell a^2}{V}\right) P\,\frac{\partial P}{\partial X} . \tag{26}$$

Boundary condition at X = 1,

$$P = 1 . \tag{27}$$

Initial condition,

$$P = P_{c_0} \quad (0 \leq X < 1),$$
$$P = 1 \quad , \quad X = 1, \tag{28}$$

where P_{c_0} is the nondimensional initial line and sensing unit pressure.

Linearization of the flow equations and the initial and boundary conditions.-- Iberall (7) obtained a solution of Equation (25) for the condition of an oscillatory reservoir pressure, and Kendall (8) solved the quasi-steady case wherein a manometer was used as the measuring device. Neither of these solutions represent the case under consideration. Ducoffe (9) obtained numerical solutions, using the method of finite differences, for three typical cases. Correlation between his experimental and analytical results was excellent. These results will be referred to later in the text. Numerical methods, while often adequate for particular cases, do not generally permit rapid estimates to be made of the response time, and therefore are of no immediate value for design purposes.

As an initial step towards obtaining an approximate solution, the problem is examined for the case of a step function in pressure which is initially small compared with the reservoir pressure. Let $\hat{p}$ be a perturbation pressure defined by

$$p_R + \hat{p} = p \ . \tag{29}$$

In nondimensional form, Equation (29) is

$$\frac{p_R + \hat{p}}{p_R} = \frac{p}{p_R} \,. \tag{30}$$

By defining $\hat{P} = \frac{\hat{p}}{p_R}$, Equation (30) is

$$1 + \hat{P} = P \,. \tag{31}$$

Substitution of Equation (31) into Equation (25) yields

$$\begin{aligned} \frac{\partial \hat{P}}{\partial \bar{t}} &= k_1 \frac{\partial}{\partial X}\left(\frac{\partial \hat{P}}{\partial X} + \hat{P}\frac{\partial \hat{P}}{\partial X} \right); \\ k_1 &= \frac{1}{8}\left(\frac{a^2}{\ell^2} \, \frac{p_R \, t_R}{\mu} \right) . \end{aligned} \tag{32}$$

Let the nondimensional perturbation pressure $\hat{P}$ be small compared with unity. That is

$$P - 1 = \hat{P} \ll 1 \,. \tag{33}$$

After the small step function has decayed into a continuous function of $\hat{P}$ versus X, the linearization process dictates the relation

$$\frac{\partial \hat{P}}{\partial X} \gg \hat{P}\frac{\partial \hat{P}}{\partial X} \,. \tag{34}$$

The term $\hat{P}\frac{\partial \hat{P}}{\partial X}$, being small compared with $\frac{\partial \hat{P}}{\partial X}$, is neglected. It will be shown later that the time lapse required to establish the condition given by Equation (34) is small. Under these assumptions, Equation (32) becomes

$$\frac{\partial \hat{P}}{\partial \bar{t}} = k_1 \frac{\partial^2 \hat{P}}{\partial X^2} \,. \tag{35}$$

The boundary condition at X = 0 (Equation (26), with Equation (31) is

$$\frac{\partial \hat{P}}{\partial \bar{t}} = k_2 (1 + \hat{P}) \frac{\partial \hat{P}}{\partial X}, \tag{36}$$

where $k_2 = \frac{\pi}{8}\left(\frac{a^2}{\ell^2}\frac{p_R\,\tau_R}{\mu}\right)\left(\frac{\ell a^2}{V}\right) = \frac{\pi \ell a^2}{V} k_1 .$

On the basis of Equation (34), $\hat{P}$ is negligible compared to unity, and Equation (36) becomes

$$\left.\frac{\partial \hat{P}}{\partial \bar{t}}\right]_{X=0} = k_2 \left.\frac{\partial \hat{P}}{\partial X}\right]_{X=0} . \tag{37}$$

The boundary condition at X = 1 becomes, in accordance with the definition of $\hat{P}$,

$$\hat{P} = 0 . \tag{38}$$

The initial condition is

$$\begin{aligned} \hat{P} &= \hat{P}_{c_0} \quad (0 \leq X < 1), \\ \hat{P} &= 0 , \quad X = 1, \end{aligned} \tag{39}$$

where $\hat{P}_{c_o}$ is the nondimensional initial perturbation pressure defined by $1 + \hat{P}_{c_o} = P_{c_o}$.

Method of solution.-- A product solution (10, 11) of Equation (35) is assumed to exist in the form

$$\hat{P} = f\,(X - 1)g\,(\bar{t}). \tag{40}$$

Equation (40) is substituted into Equation (35) and the variables are separated; the equation of motion becomes

$$\frac{\dot{g}}{g} = k_1 \frac{f''}{f} = \beta, \tag{41}$$

where the dots and primes denote differentiation with respect to time and spatial position, respectively, and where β is a constant. The two resulting ordinary differential equations are

$$\frac{dg}{g} = \beta d\bar{t} \tag{42}$$

and

$$\frac{d^2 f}{dX^2} - \frac{\beta}{k_1} f = 0. \tag{43}$$

The solution of Equation (42) is

$$g(\bar{t}) = c_1 e^{\beta \bar{t}}, \tag{44}$$

where c_1 is an arbitrary constant of integration. Since the pressure is a decreasing function of time, β must be negative. To emphasize this, let $\beta = -\gamma^2$, where γ is real.
Then

$$g(\bar{t}) = c_1 e^{-\gamma^2 \bar{t}}. \tag{45}$$

The solution of Equation (43) is

$$f(X-1) = c_2 \cos\sqrt{\frac{\gamma^2}{k_1}}(X-1) + c_3 \sin\sqrt{\frac{\gamma^2}{k_1}}(X-1), \tag{46}$$

where the c's are arbitrary constants. The expression for nondimensional perturbation pressure, $\hat{P}$, in Equation (40) is given by

$$\hat{P} = e^{-\gamma^2 \bar{t}}\left[c_4 \cos\sqrt{\frac{\gamma^2}{k_1}}(X-1) + c_5 \sin\sqrt{\frac{\gamma^2}{k_1}}(X-1)\right]. \tag{47}$$

Now $\hat{P} = 0$ at $X = 1$. Hence,

$$0 = c_4 e^{-\gamma^2 \bar{t}}, \tag{48}$$

and $c_4=0$.

Equation (47), with the boundary condition at X=0, becomes

$$c_5 \gamma^2 e^{-\gamma^2 \bar{t}} \sin\sqrt{\frac{\gamma^2}{k_1}} = c_5 k_2 \sqrt{\frac{\gamma^2}{k_1}}\, e^{-\gamma^2 \bar{t}} \cos\sqrt{\frac{\gamma^2}{k_1}}. \tag{49}$$

By definition of k_1 and k_2 and since $c_5 = 0$, Equation (49) becomes

$$\lambda \tan\lambda = \frac{\pi \ell a^2}{V}, \tag{50}$$

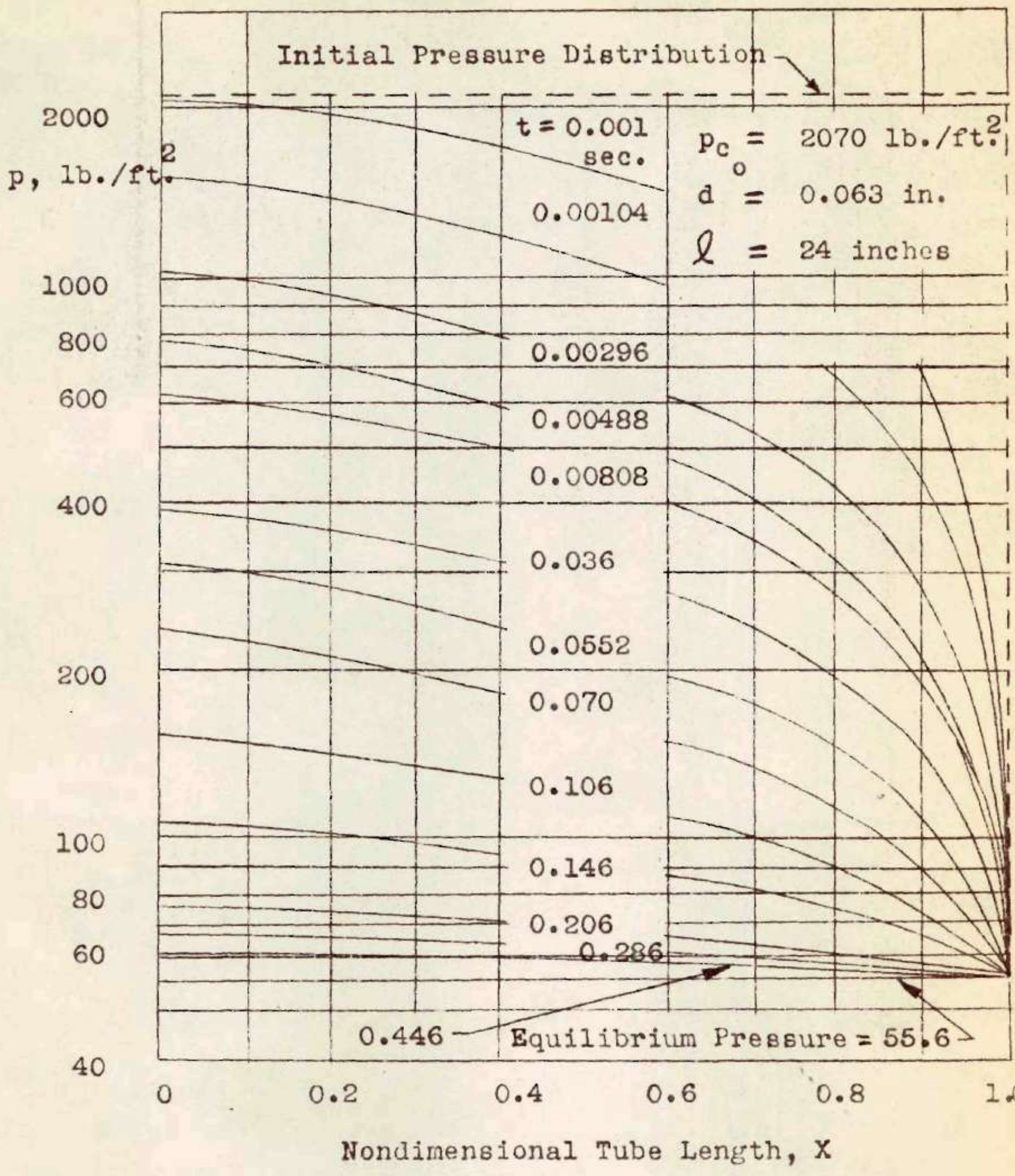
where $\lambda = \sqrt{\frac{\gamma^2}{k_1}}$. Values of λ given by Equation (50) are merely the z- components of the intersections of the hyperbola $y\ z = \pi \ell a\ 2/V$ and the tangent curve $y = \tan z$. This is discussed later in detail.

At this stage certain analytical results are considered. Figure 4 shows some typical line pressure distributions obtained by Ducoffe (12) in the manner previously described. For such a typical case where the initial line-capsule perturbation pressure is large compared with the reservoir pressure, the step function decays very rapidly during the early portion of the transient period. In addition to this decay of the step into a continuous function, the capsule pressure falls very rapidly during this period. It may be expected, then, that after a short period the linear equations may be applied with a fair degree of accuracy.

Obviously a one-term solution such as

$$\hat{P} = c_5 e^{-\gamma^2 \bar{t}} \sin \sqrt{\frac{\gamma^2}{k_1}} (X-1) \tag{51}$$

cannot be made to satisfy the initial condition of a step function in pressure. The step function could be represented by an infinite series of such terms, but numerical methods would be required to approximate the response time from the resulting exponential equation. However, since it is desired to utilize a linearized solution for the case where a large step function has decayed, it is more accurate to satisfy an initial condition which approximates the dissipated step function. Although a step of small strength dissipates more slowly than does one of large strength, it is believed that for such a case it is unnecessary to satisfy the step function initial condition to achieve reasonable accuracy.



Typical Spatial Pressure Distribution

Figure 4

Thus, for the case where the initial pressure differential is small compared to the reservoir pressure, the initial condition is taken as an approximation to the dissipated step function in the following manner: At $X = 0$ and when $\bar{t} = o$, $\hat{P} = P_{c_o} - 1 = \hat{P}_{c_o}$.

For this condition Equation (51) becomes

$$\hat{P}_{c_o} = -c_5 \sin\sqrt{\frac{\gamma^2}{k_1}}\,, \tag{52}$$

and hence

$$\hat{P} = \hat{P}_{c_o} \frac{\sin\sqrt{\frac{\gamma^2}{k_1}}(1-X)}{\sin\sqrt{\frac{\gamma^2}{k_1}}} e^{-\gamma^2\bar{t}}. \tag{53}$$

It was previously noted that the parameter λ is multi-valued. To avoid the existence of a zero perturbation pressure on the interval $0 \leq X < 1$, the smallest value of λ (i.e., $0 < \lambda < \frac{\pi}{2}$) is chosen. If the next smallest value of λ were chosen, the period $\frac{2\pi}{\lambda}$ of the functions in $\lambda(1-X)$ would be less than 2, since this value of λ is in the range $\pi < \lambda < \frac{3\pi}{2}$. If the period were less than 2, at least one zero perturbation pressure would exist on the interval $0 \leq X < 1$ because of the fixed end conditions $\hat{P}(0)\big]_{\bar{t}=0} = \hat{P}_{c_o}$ and $\hat{P}(1) = 0$.

The nondimensional capsule perturbation pressure is denoted by $\hat{P}_c$, and Equation (53) becomes, at the capsule, $(X = o)$,

$$\hat{P}_c = \hat{P}_{c_o} e^{-\gamma^2\bar{t}}. \tag{54}$$

From the definition of the response time, it is seen that $\hat{P}_c = 0.01$ when $\bar{t} = 1$, and therefore

$$\gamma^2 = -\ln\left(\frac{0.01}{\hat{P}_{c_0}}\right). \tag{55}$$

But $\gamma^2 = k, \lambda^2 = \frac{1}{8}\left(\frac{a^2}{\ell^2}\frac{P_R\,\tau_R}{\mu}\right)\lambda^2$, and therefore

$$\tau_R = -8\left(\frac{\ell}{a\lambda}\right)^2\frac{\mu}{P_R}\ln\left(\frac{0.01}{\hat{P}_{c_0}}\right), \tag{56}$$

where τ_R is the approximate response time for the case $\hat{P}_{c_0} << 1$.

In dimensional form, Equation (53) is

$$P = P_R + (P_{c_0} - P_R)\frac{\sin\frac{\lambda}{\ell}(\ell - x)}{\sin\frac{\lambda}{\ell}}\, e^{-\frac{1}{8}\left(\frac{a^2}{\ell^2}\frac{P_R}{\mu}\lambda^2\right)t}, \tag{57}$$

and similarly, Equation (56) is

$$\tau_R = -8\left(\frac{\ell}{a\lambda}\right)^2\frac{\mu}{P_R}\ln\left(\frac{0.01\,P_R}{P_{c_0} - P_R}\right). \tag{58}$$

Equation (57) becomes, at the pressure sensing end,

$$P_c = P_R + (P_{c_0} - P_R)\, e^{-\frac{1}{8}\left(\frac{a^2}{\ell^2}\frac{P_R}{\mu}\lambda^2\right)t}. \tag{59}$$

It should be noted that, where $\hat{P}_{c_0} << 1$, Equation (57) approximates a quasi-initial condition reached at a finite period of time after the beginning of the run. This time lapse is extremely small and is neglected.

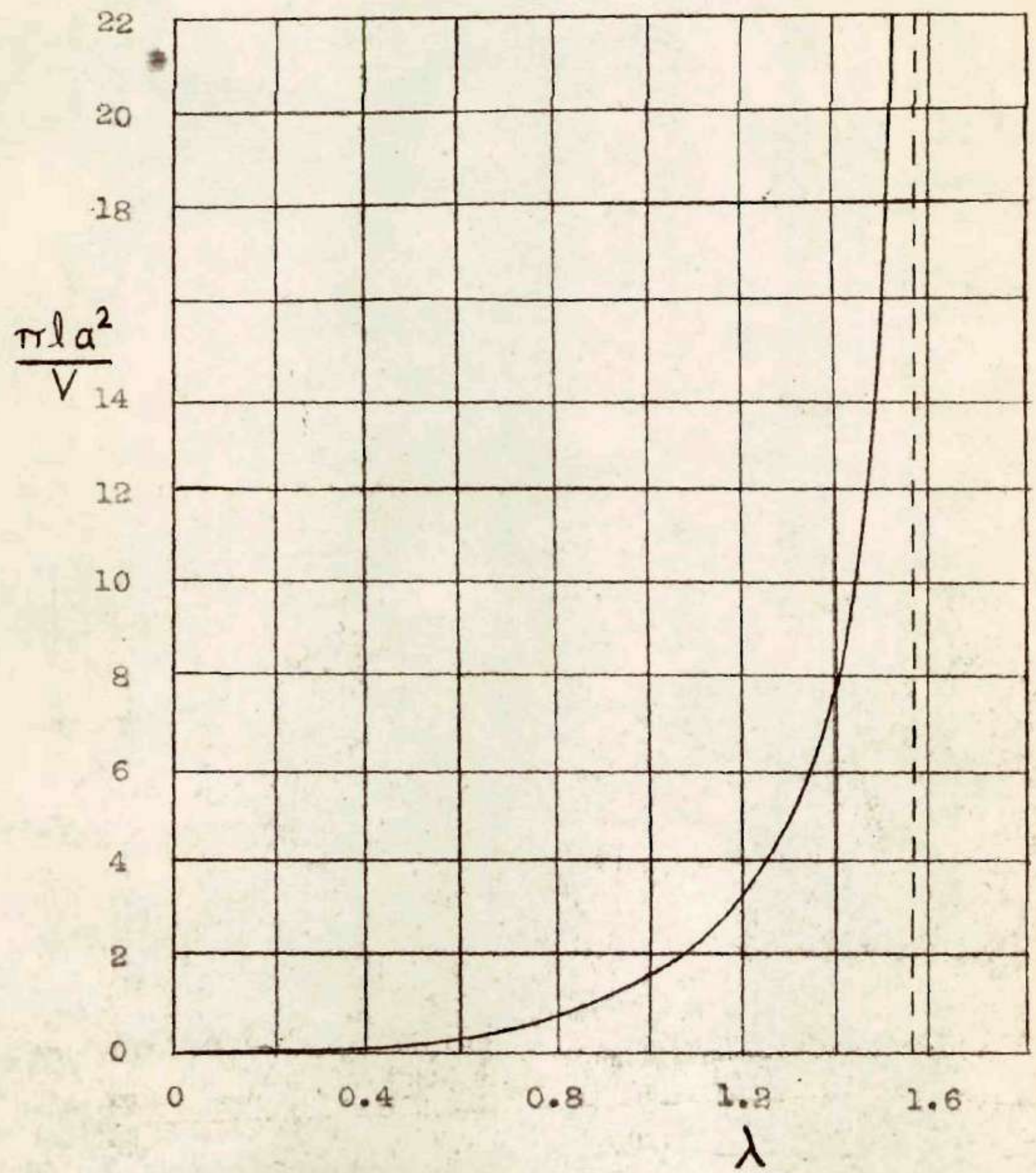
CHAPTER III

DISCUSSION OF RESULTS

General -- Examination of Equation (58) shows that the diameter and length of the capillary tube and the reservoir pressure have the most critical effects on the response time. The nature of λ in Equation (50) does not permit the response time to be expressed explicitly in terms of the diameter and length of the capillary tube and the volume of the pressure capsule. To assist in determining the effects of ℓ, a, and V, the parameter λ is shown in Figure 5 as a function of the geometric similarity parameter $\pi \frac{\ell a^2}{V}$. Figure 6 shows λ as a function of tube length for several standard diameters with a capsule reservoir of 0.106 cubic inches. This is the volume used by Ducoffe (13) in most of his experiments, and is used here since the results of this analysis are compared to his experimental and analytical results for these cases.

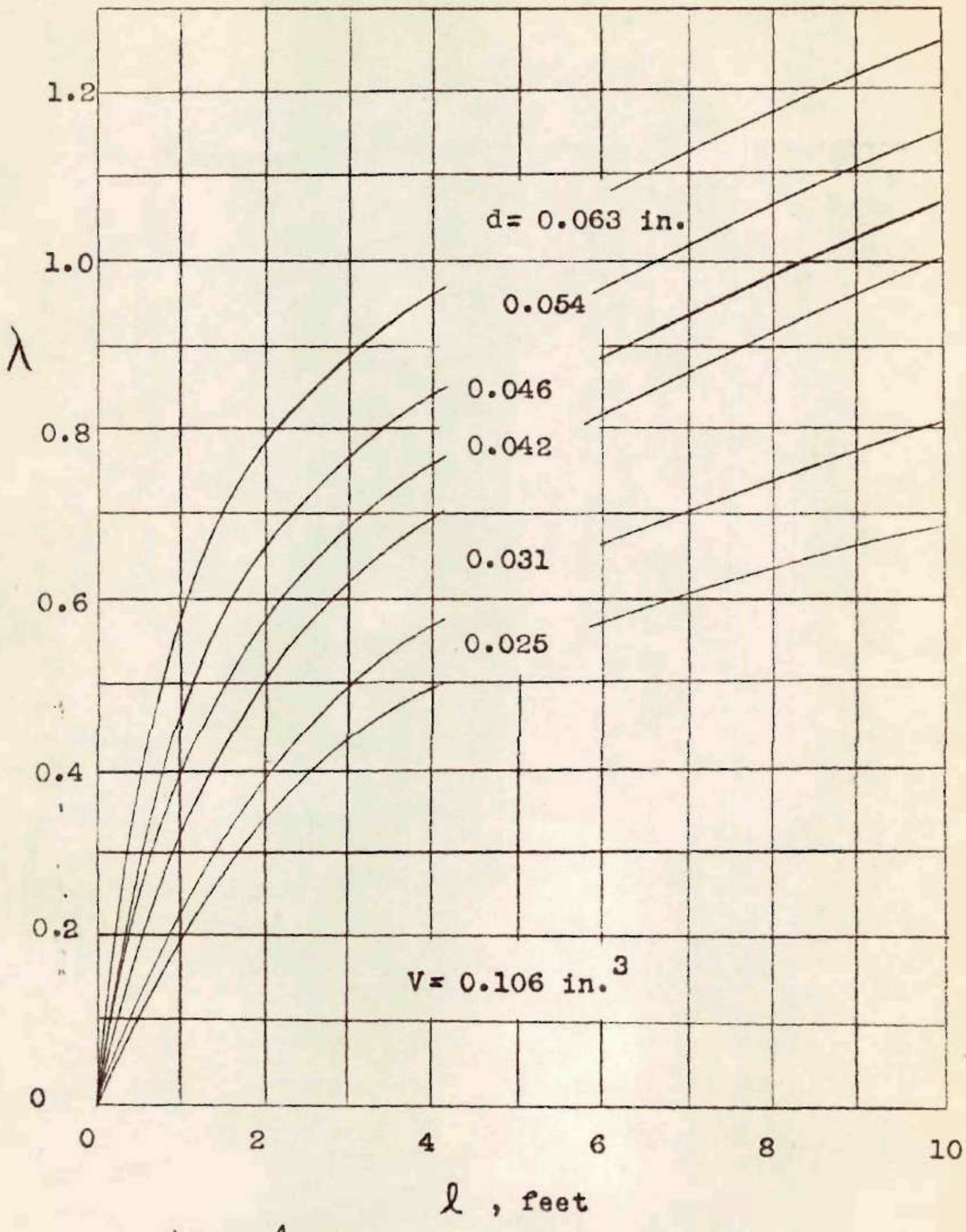
Comparison of theory with experiment. -- It was mentioned previously that Ducoffe (14) obtained numerical solutions to the set of equations (25), (26), (27), and (28) for three typical cases. He established the validity of the set of equations by conducting experimental investigations in which capsule pressure as a function of time was recorded.

The three cases are:



λ vs. Geometric Similarity Parameter

Figure 5



λ vs. l , For Various Tube Diameters

Figure 6

Case I:

$p_{c_o} = 2070$ lbs./ft^2

$p_R = 55.6$ lbs./ft^2

$a = 0.0315$ inch

$\ell = 2$ feet

$V = 0.106$ cubic inch

$\mu = 3.82 \times 10^{-7}$ slugs/ft.-sec.

Case II:

$p_{c_o} = 2070$ lbs./ft^2

$p_R = 55.6$ lbs./ft^2

$a = 0.0125$ inch

$\ell = 2$ feet

$V = 0.106$ cubic inch

$\mu = 3.82 \times 10^{-7}$ slugs/ft.-sec.

Case III:

$p_{c_o} = 2090$ lbs./ft^2

$p_R = 1414$ lbs./ft^2

$a = 0.023$ inch

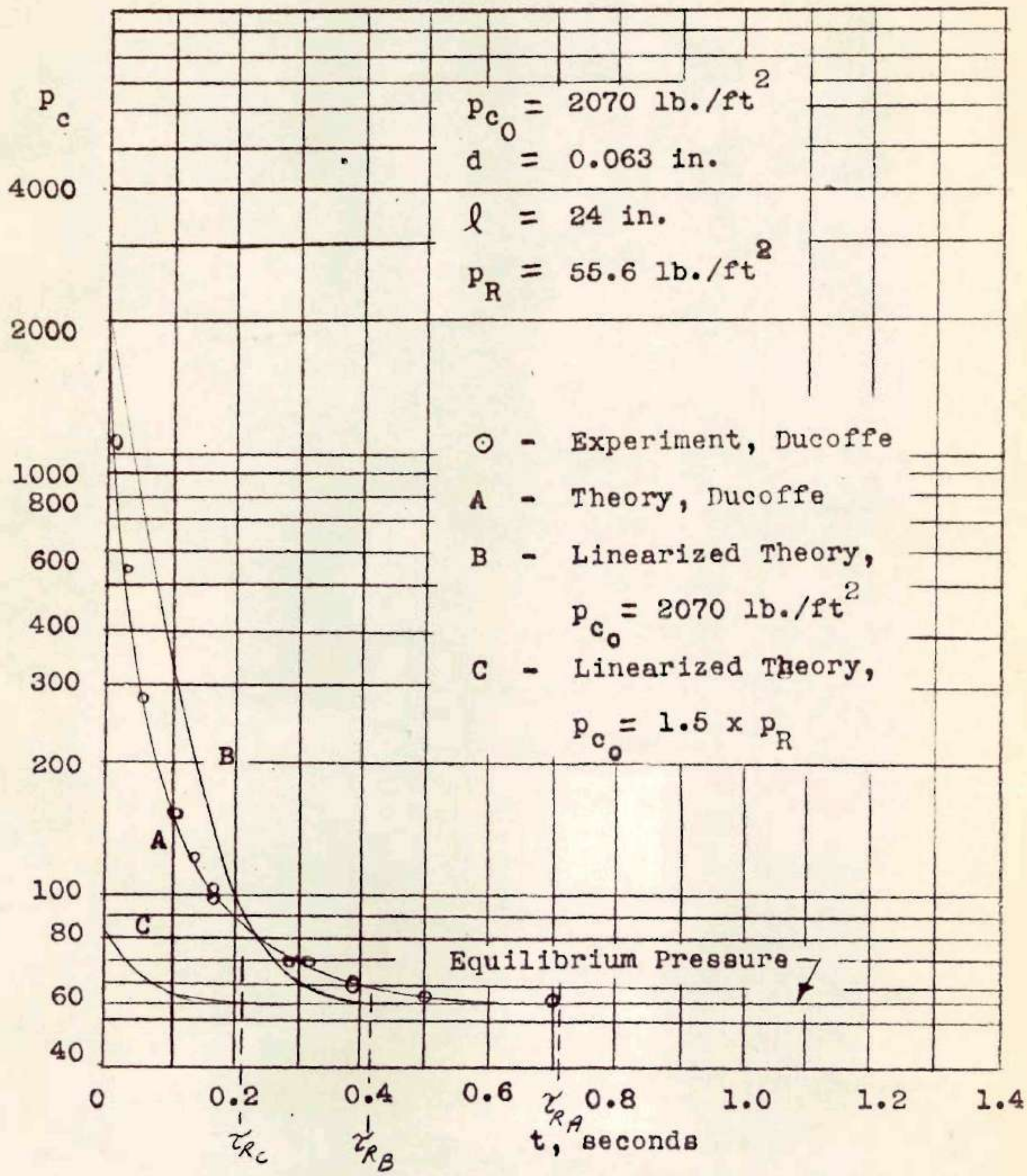
$\ell = 2$ feet

$V = 0.6362$ inch^3

$\mu = 3.788 \times 10^{-7}$ slugs/ft.-sec.

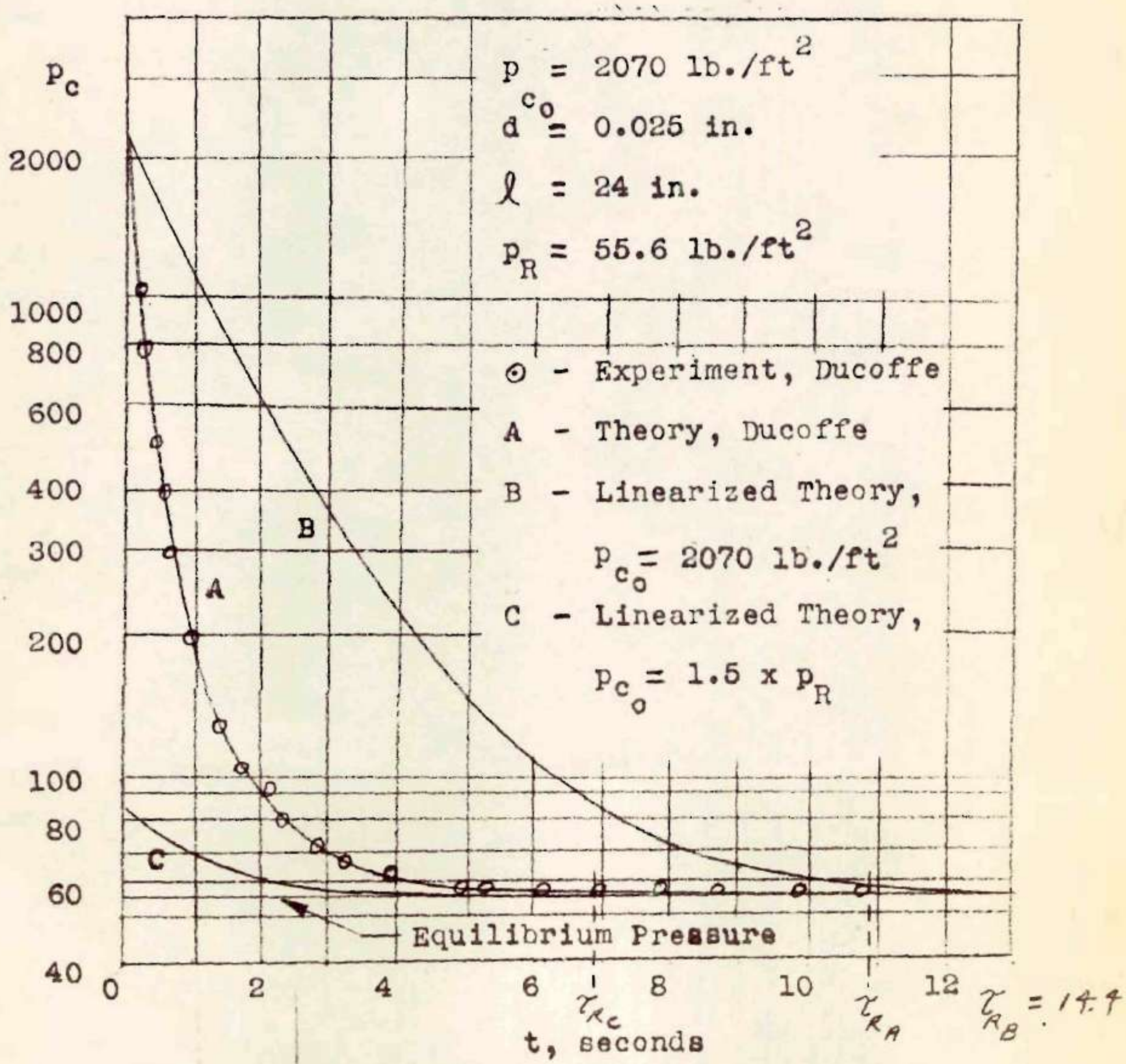
Both the results of Reference 9 and the results of this analysis are presented in Figures 7, 8, and 9 for cases I, II, and III, respectively. Circled points and the curves designated "A" are respectively the experimental and theoretical reference results. Curves designated "B" were obtained by substitution of the actual initial line-capsule pressure into Equation (59). The response times so obtained for cases I, II, and III are 0.42, 14.4, and 0.19 seconds, respectively, while the respective reference results are 0.71, 10.9, and 0.38 seconds. Such substitutions violate the condition which permitted linearization of the equations, that is, the initial pressure step functions are large compared with the reservoir pressures. In view of this fact, curves "B" agree reasonably well with the reference results. The error in cases I and III makes no appreciable difference, since the response time is extremely small. However, for Case II, the error of thirty-two per cent is not negligible. It is believed that substitution of the actual initial pressure of the line and capsule into Equation (59) gives results which can be of qualitative interest at best.

Curves "C" in Figures 7 and 8 were obtained by substituting $p_{c_o} = 1.5\ p_R$ into Equation (59). The respective response timer obtained in this manner are 0.21 and 6.9 seconds. Except for a time displacement toward the origin, these curves closely approximate the experimentally obtained ones. This time displacement is merely the time lapsed prior to the quasi-



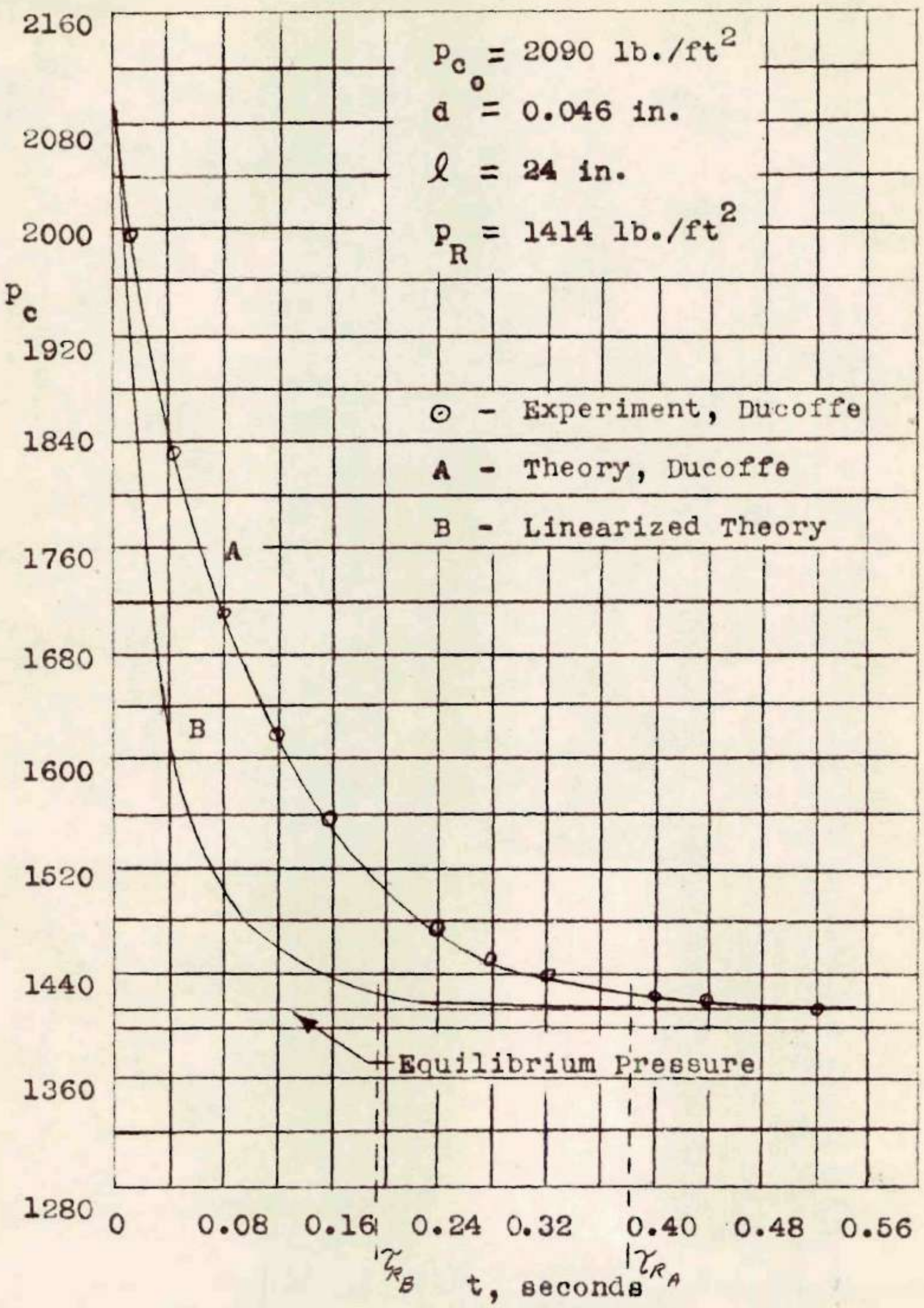
Pressure Variation In The Capsule Reservoir

Figure 7



Pressure Variation In The Capsule Reservoir

Figure 8



Pressure Variation In The Capsule Reservoir

Figure 9

initial condition $p_c = 1.5\ p_R$. Since little is known concerning the process during the early transient states, a correction factor for this time lapse would be essentially empirical. Because of the limited amount of experimental data available, no attempt is made here to predict this time lapse as a fraction of total response time. However, these results demonstrate that the linear equations represent the process fairly closely after the perturbation has dissipated. This lends support to the belief that the linear equation may be applied with reasonable accuracy if the initial pressure step function is small. Unfortunately, no experimental data is available for comparison for the cases where the initial step functions are initially small.

It should be noted that, for the above three cases, the linear theory underestimates the small response times and overestimates the larger one. For the cases where the linearization process is permissible, it may be expected that the linear equations will slightly underestimate the response time for two reasons. The first is that the time required for the step function to decay into a continuous function is neglected. The second reason is seen by examining Equation (32). The term $\frac{\partial}{\partial X}\left(\hat{P}\frac{\partial \hat{P}}{\partial X}\right)$, which is neglected, is positive, thus making $\hat{P}$ decrease more rapidly with time.

CHAPTER IV

CONCLUSIONS

1. If the nondimensional perturbation pressure $\frac{p_{c0} - p_R}{p_R}$ is small compared to unity, Equation (58) may be expected to accurately predict the response time of the idealized system.

2. On the basis of Figures 7 and 8, where $\frac{p_{c0} - p_R}{p_R}$ is not small compared with unity, Equation (58) will not accurately predict the response time unless allowance is made for the time required for the equation to become applicable.

3. In order to minimize response time, the capillary tube length, the volume of the pressure sensing device, and the initial line-capsule pressure should be minimized, whereas the capillary tube diameter should be as large as possible.

4. The advantage of previous pump-down of capsule pressure becomes more pronounced as $\hat{P}_c \longrightarrow 0$, and is negligible outside the pressure range for which the linear equations apply.

CHAPTER V

RECOMMENDATIONS

1. An attempt should be made to obtain a solution which more closely approximates the initial step function by utilizing a finite number of terms of the form

$$\hat{P}_{c_0} \frac{\sin \lambda (1-X)}{\sin \lambda} e^{-k_1 \lambda^2 \bar{t}} .$$

2. An experimental investigation should be conducted for $\hat{P}_{c_0} << 1$ to check Equation (58) and the possible results of the first recommendation.

3. An attempt should be made to solve the linearized equation of motion for the system utilizing a fluid manometer as a pressure measuring device. Such a solution should be compared to the results obtained in Reference 15.

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